3. In free space the potential is

$$V = y^2 z \ln x.$$

Find out whether V satisfies Laplace's equation.[5]† Then find the total charge within the region defined by $1 < x < e$, $-1 < y < 0$ and $0 < z < 1$, where e is the natural number. [15]

Solution.

$$\begin{aligned}\nabla^2 V &= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \\ &= -\frac{1}{x^2} y^2 z + 2z \ln x = \neq 0\end{aligned}$$

Therefore, V does not satisfy Laplace's equation.

#

Poisson's;

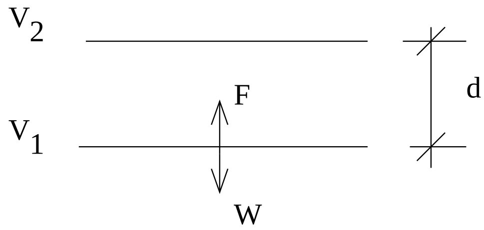
$$\begin{aligned}\nabla^2 V &= -\frac{\rho_v}{\epsilon_0} \\ \rho_v &= \epsilon_0 \left(\frac{y^2 z}{x^2} - 2z \ln x \right)\end{aligned}$$

$$\begin{aligned}Q &= \epsilon_0 \int_0^1 \int_{-1}^0 \int_1^e \left(\frac{y^2 z}{x^2} \right) \mathrm{d}x \mathrm{d}y \mathrm{d}z \\ &= \epsilon_0 \int_0^1 \int_{-1}^0 \left(-\frac{y^2 z}{x} - 2z(x \ln x - x) \right| 1^e \mathrm{d}y \mathrm{d}z \\ &= \epsilon_0 \int_0^1 \int_{-1}^0 \left(y^2 z \left(1 - \frac{1}{e} \right) - 2z(e+1) - e \right) \mathrm{d}y \mathrm{d}z \\ &= \epsilon_0 \int_0^1 \left(\left(1 - \frac{1}{e} \right) \frac{y^3 z}{3} - 2z(e+1)y - ey \right| -1^0 \mathrm{d}z \\ &= \epsilon_0 \int_0^1 \left(\left(1 - \frac{1}{e} \right) \frac{1}{3} z - 2z(e+1) - e \right) \mathrm{d}z \\ &= \epsilon_0 \int_0^1 \left(\left(\frac{1}{3} - \frac{1}{3e} - 2e - 2 \right) z - e \right) \mathrm{d}z \\ &= \epsilon_0 \int_0^1 \left(\frac{e - 1 - 6e^2 - 6e}{3e} z - e \right) \mathrm{d}z \\ &= -\epsilon_0 \int_0^1 \left(\frac{6e^2 + 5e + 1}{3e} z - e \right) \mathrm{d}z \\ &= -\epsilon_0 \left(\frac{6e^2 + 5e + 1}{6e} z^2 - ez \right| 0^1 \\ &= -\epsilon_0 \left(\frac{6e^2 + 5e + 1}{6e} - e \right) = -\frac{5e+1}{6e} \epsilon_0\end{aligned}$$

#

† Numbers in square brackets are points of their corresponding tasks.

4. An electrometer is shown in the following picture.



Show that [16]

$$\Delta V = V_1 - V_2 = \left[\frac{2Fd^2}{\epsilon S}\right]^{1/2}$$

When in free space ΔV measured is 5V. What would the ΔV be if the plates is filled with polystyrene whose ϵ_r is 2.55.[4] Here V_1 and V_2 are the potentials of plate 1 and respectively plate 2, F the force measured in terms of weight, W the weight used in measuring the force, S the area of each plate, ϵ the permittivity and d the distance between the two plates as shown.

Solution.

$$\begin{aligned}\mathbf{E} &= \frac{\rho_s}{2\epsilon}\mathbf{a}_n + \frac{-\rho_s}{2\epsilon}(-\mathbf{a}_n)\\ &= \frac{\rho_s}{\epsilon}\mathbf{a}_n\end{aligned}$$

$$E = \frac{\rho_s}{\epsilon}$$

$$\mathbf{E} = -\nabla V$$

$$\frac{\partial V}{\partial z} = \frac{\mathrm{d}V}{\mathrm{d}z} = -\frac{\rho_s}{\epsilon}$$

$$\int_{V_1}^{V_2} \mathrm{d}V = -\int_0^d \frac{\rho_s}{\epsilon}\mathrm{d}z$$

$$V_1 - V_2 = \frac{\rho_s d}{\epsilon}$$

$\mathbf{F} = Q\mathbf{E}$ and $Q = \rho_s S$.

$$\mathbf{F} = \rho_s S \cdot \frac{\rho_s}{2\epsilon}\mathbf{a}_n = \frac{\rho_s^2 S}{2\epsilon}\mathbf{a}_n$$

$$F = \frac{\rho_s^2 S}{2\epsilon}$$

$\rho_s = \left(\frac{2F\epsilon}{S}\right)^{1/2}$;

$$V_1 - V_2 = \left(\frac{2F\epsilon}{S}\right)^{1/2}\frac{d}{\epsilon} = \left[\frac{2Fd^2}{\epsilon S}\right]^{1/2}$$

Shown.

#

$$\Delta V = \frac{K}{\epsilon_r\epsilon_0}$$

$\epsilon_r = 1$;

$$\Delta V_1 = \frac{K}{\epsilon_0} = 5 \quad \Rightarrow \quad K = 5\epsilon_0$$

$\epsilon_r = 2.55$;

$$\begin{aligned}\Delta V_2 &= \frac{K}{2.55\epsilon_0} = \frac{5\epsilon_0}{2.55\epsilon_0}\\ &= 1.96 \quad \mathrm{V}\end{aligned}$$

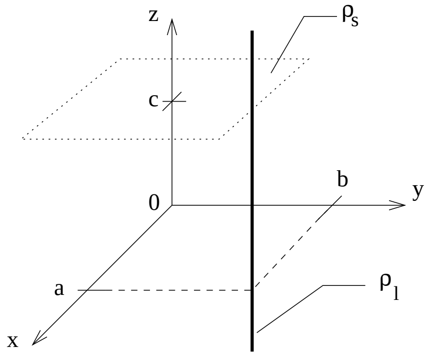
#

5. In a dielectric where $\epsilon = \epsilon_r \epsilon_0$ a line $x = a$, $y = b$ carries a charge $\rho_l \mathrm{C} \cdot \mathrm{m}^{-1}$ while a plane $z = c$ carries a charge $\rho_S \mathrm{C} \cdot \mathrm{m}^{-2}$. Sketch the picture.[2]

Then find the following.

a. $\mathbf{E}_l$ due to the line [5]
b. $\mathbf{E}_S$ due to the plane [5]
c. The force on a point charge q_p at the origin [5]
d. With the existing configuration, would it be possible to find $\rho_l \neq 0$ such that $\mathbf{E}(\mathbf{0}) = 0$?[1] Why, or why not?[2]

Solution.



$$\mathbf{r} + 0 = (0, 0, 0)$$

infinite line;

$$\mathbf{E}_l = \frac{\rho_l}{2\pi\epsilon\rho}\mathbf{a}_\rho$$

infinite plane;

$$\mathbf{E}_s = \frac{\rho_s}{2\epsilon}\mathbf{a}_n$$

$\rho = \sqrt{a^2 + b^2}$;

$$\begin{aligned}\vec{\rho} &= \mathbf{r}_0 - \mathbf{r}_\rho \\ &= (0,0,0) - (a,b,0) = (-a,-b,0) \\ \mathbf{a}_\rho &= \frac{(-a,-b,0)}{\sqrt{a^2+b^2}}\end{aligned}$$

for S;

$$\mathbf{a}_n = -\mathbf{a}_z$$

a.

$$\mathbf{E}_l = \frac{\rho_l}{2\pi\epsilon\sqrt{a^2+b^2}} \frac{(-a,-b,0)}{\sqrt{a^2+b^2}} = \frac{\rho_l(-a,-b,0)}{2\pi\epsilon(a^2+b^2)} \quad \mathrm{V \cdot m^{-1}}$$

#

b.

$$\mathbf{E}_s = \frac{\rho_s}{2\epsilon}(-\mathbf{a}_z) = -\frac{\rho_s}{2\epsilon}\mathbf{a}_z = -\frac{\rho_s}{2\epsilon}(0,0,1) \quad \mathrm{V \cdot m^{-1}}$$

#

c.

$$\mathbf{E} = \mathbf{E}_l + \mathbf{E}_s = \frac{\rho_l(-a,-b,0)}{2\pi\epsilon(a^2+b^2)} - \frac{\rho_s}{2\epsilon}(0,0,1) \quad \mathrm{V \cdot m^{-1}}$$

$$\begin{aligned}\mathbf{F} = Q\mathbf{E} &= q_p\left[\frac{\rho_l(-a,-b,0)}{2\pi\epsilon(a^2+b^2)} - \frac{\rho_s}{2\epsilon}(0,0,1)\right] \\ &= -\frac{q_p}{2\epsilon}\left[\frac{\rho_l}{\pi(a^2+b^2)}(a\mathbf{a}_x + b\mathbf{a}_y) + \rho_s\mathbf{a}_z\right] \quad \mathrm{V \cdot m^{-1}}\end{aligned}$$

#

d. For $\rho_l \neq 0$ and $\rho_s \neq 0$;

$$\frac{\rho_l(-a,-b,0)}{2\pi\epsilon(a^2+b^2)} \neq \frac{\rho_s}{2\epsilon}(0,0,1)$$

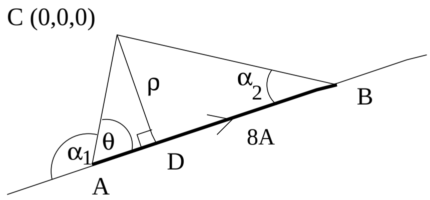
Therefore it is not possible to find $\rho_l \neq 0$ such that $\mathbf{E}$ at the origin is zero.

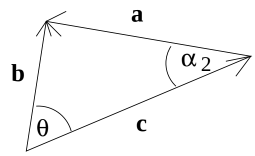
#

6. A straight line of conductor connects A$(7, 1, -2)$ to B$(-3, 5, 9)$. There is an electric current which flows from A to B and its value is 8 A. Find the magnetic field $\mathbf{H}$ due to the conductor AB at the origin.[15] If the line extends infinitely in both directions, what becomes the magnetic field?[3] Compare this case with ours one where the length of the conductor is finite.[2]

Solution.

$$\mathbf{H} = \frac{I}{4\pi\rho}(\cos\alpha_2 - \cos\alpha_1)\mathbf{a}_\phi$$





$$\mathbf{c} = \mathbf{B} - \mathbf{A} = (-3, 5, 9) - (7, 1, -2) = (-10, 4, 11)$$

$$\begin{aligned} C &= \sqrt{10^2 + 4^2 + 11^2} = \sqrt{237} \\ a &= \sqrt{7^2 + 1^2 + 4} = \sqrt{54} \\ b &= \sqrt{9 + 25 + 81} = \sqrt{115} \end{aligned}$$

law of cosine;

$$\begin{aligned} b^2 &= a^2 + c^2 - 2ac\cos\alpha_2 \\ 115 &= 54 + 237 - 2\sqrt{54(237)}\cos\alpha_2 \\ \cos\alpha_2 &= 0.7779 \\ \alpha_2 &= 38.93^\circ \end{aligned}$$

similarly;

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\theta \\ 54 &= 115 + 237 - 2\sqrt{115(237)}\cos\theta \\ \cos\theta &= 0.9025 \\ \theta &= 25.51^\circ \\ \alpha_1 &= 180^\circ - 25.52^\circ = 154.49^\circ \\ \cos\alpha_1 &= -0.9025 \end{aligned}$$

let

$$\begin{aligned} r &= \frac{\mathbf{b} \cdot \mathbf{c}}{c^2} \\ \mathbf{b} &= (0, 0, 0) - (7, 1, -2) = (-7, -1, 2) \\ r &= \frac{(-7, -1, 2)(-10, 4, 11)}{237} = 0.3713 \end{aligned}$$

$0 < r < 1 \quad \Rightarrow \quad$ point D is on $\mathbf{c}$.

$$
\begin{aligned}
D &= (7, 1, -2) + 0.3713(-10, 4, 11) \\
&= (3.29, 2.49, 2.08) \\
\rho &= \sqrt{3.29^2 + 2.49^2 + 2.08^2} = 4.62 \\
\mathbf{a}_\phi &= \mathbf{a}_l \times \mathbf{a}_\rho
\end{aligned}
$$

Let the directional cosines of $\mathbf{a}_l$ be $\cos\alpha_l$, $\cos\beta_l$ and $\cos\gamma_l$, and those of $\mathbf{a}_\rho$ be $\cos\alpha_\rho$, $\cos\beta_\rho$ and $\cos\gamma_\rho$. Then,

$$
\begin{aligned}
\cos\alpha_l &= \frac{-10}{\sqrt{237}} = -0.6496 \\
\cos\beta_l &= \frac{4}{\sqrt{237}} = 0.2598 \\
\cos\gamma_l &= \frac{11}{\sqrt{237}} = 0.7145
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{a}_l &= \sqrt{237}(-0.6496, 0.2598, 0.7145) \\
\vec{\rho} &= -\mathbf{d} = (-3.29, -2.49, -2.08) \\
\cos\alpha_\rho &= -\frac{3.29}{4.62} = -0.7121 \\
\cos\beta_\rho &= -\frac{2.49}{4.62} = -0.5390 \\
\cos\gamma_\rho &= -\frac{2.08}{4.62} = -0.4502 \\
\mathbf{a}_\rho &= 4.62(-0.7121, -0.5390, -0.4502)
\end{aligned}
$$

#

$$
\begin{aligned}
\mathbf{a}_\phi &= \mathbf{a}_l \times \mathbf{a}_\rho \\
&= 4.62\sqrt{237}\left((-0.6496, 0.2598, 0.7145) \times (-0.7121, -0.539, -0.4502)\right) \\
&= 4.62\sqrt{237}\left((0.2598)(-0.4502) - (0.7145)(-0.539), (0.7145)(-0.7121) - (-0.6496)(-0.4502), (-0.6496)(-0.539) - (0\right. \\
&= 4.62\sqrt{237}(0.2682, -0.8012, 0.5351) \\
&= (19.08, -56.98, 38.06)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{H} &= \frac{8}{4\pi 4.62}\left(0.7779 - (-0.9025)\right)(19.08, -56.98, 38.06) \\
&= (4.42, -13.19, 8.81) \quad \mathrm{A \cdot m^{-1}}
\end{aligned}
$$

#

If the line becomes infinitely long, then independently of the geometry $\alpha_2 = 0°$ and $\alpha_1 = 180°$.

$$
\begin{aligned}
\mathbf{H} &= \frac{I}{2\pi\rho}\mathbf{a}_\phi \\
&= \frac{8}{2\pi 4.62}(19.08, -56.98, 38.06) \\
&= (5.26, -15.70, 10.49) \quad \mathrm{A \cdot m^{-1}}
\end{aligned}
$$

#

$$
\begin{aligned}
H_1 &= \sqrt{4.42^2 + 13.19^2 + 8.81^2} = 16.47 \\
H_2 &= \sqrt{5.26^2 + 15.7^2 + 10.49^2} = 19.60
\end{aligned}
$$

#

$\mathbf{H}$ of the infinite line is stronger by $(19.6 - 16.47)100/16.47 = 19$ per cent.

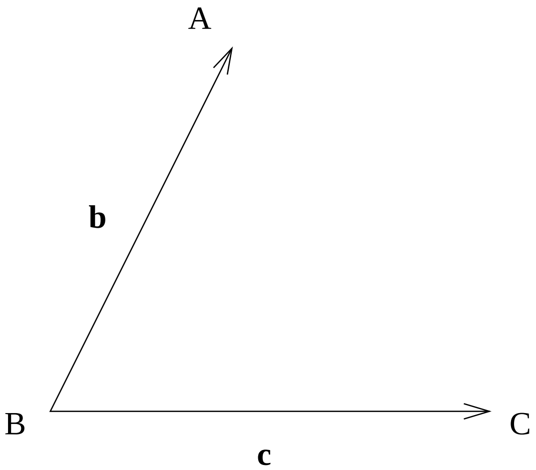
#

7. Answer the following questions on vector analysis.

i. Three points, namely A(3, 5, 8), B(4, 7, 2) and C(1, 9, 4), lie on a plane the charge density of which is $6\epsilon_0 \mathrm{C} \cdot \mathrm{m}^{-2}$. Find the unit vector $\mathbf{a}_n$ normal to the plane.[11] Find further $\mathbf{E}$ and $\mathbf{D}$.[4]

ii. Given a vector A(ayz, bxz, cxy) where a, b and c are constants, show that it is solenoidal.[2] If in addition we know that it is also irrotational, what must the values of a, b and c be?[3]

Solution. Let



then

$$\mathbf{b} = (4, 7, 2) - (3, 5, 8) = (1, 2, -6)$$
$$\mathbf{c} = (1, 9, 4) - (3, 5, 8) = (-2, 4, -4).$$

The vector normal to the plane;

$$\begin{aligned} \mathbf{N} &= \mathbf{b} \times \mathbf{c} \\ &= ((-8 + 24), (12 + 4), (4 + 4)) \\ &= (16, 16, 8) \end{aligned}$$

$$\mathbf{a}_n = \frac{(16, 16, 8)}{\sqrt{16^2 + 16^2 + 8^2}} = \frac{1}{24}(16, 16, 8) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

#

$$\mathbf{E} = \frac{\rho_s}{2\epsilon_0}\mathbf{a}_n = \frac{6\epsilon_0}{2\epsilon_0}\mathbf{a}_n = 3\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = (2, 2, 1) \quad \mathrm{V} \cdot \mathrm{m}^{-1}$$

#

$$\begin{aligned} \mathbf{D} &= \frac{\rho_s}{2}\mathbf{a}_n = \frac{6\epsilon_0}{2}\mathbf{a}_n \\ &= 3\epsilon_0\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right) = \epsilon_0(2, 2, 1) \quad \mathrm{C} \cdot \mathrm{m}^{-2} \end{aligned}$$

#

solenoidal;

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}A_x + \frac{\partial}{\partial y}A_y + \frac{\partial}{\partial z}A_z = 0$$

irrotational;

$$\nabla \times \mathbf{A} = \left[\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right]\mathbf{a}_x + \left[\frac{\partial}{\partial z}A_x - \frac{\partial}{\partial x}A_z\right]\mathbf{a}_y + \left[\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right]\mathbf{a}_z = 0$$

$$\mathbf{A} = (ayz, bxz, cxy)$$

Since $\nabla \cdot \mathbf{A} = 0$, therefore $\mathbf{A}$ is solenoidal.

#

if **A** irrotational;

$$cx - bx = 0$$
$$ay - cy = 0$$
$$bz - az = 0$$

that is,

$$c - b = 0$$
$$a - c = 0$$
$$b - a = 0$$

in other words;

$$a = b = c$$

#